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EVERY NULL-ADDITIVE SET IS MEAGER-ADDITIVE

BY

SAHARON SHELAH*

Institute of Mathematics, The Hebrew University of Jerusalem
Jerusalem 91904, Israel, e-mail: shelah@math.huji.ac.il

ABSTRACT

It is proved that every null-additive subset of ${}^{\omega}2$ is meager-additive. Several characterizations of the null-additive subsets of ${}^{\omega}2$ are given, as well as a characterization of the meager additive subsets of ${}^{\omega}2$. Under CH, an uncountable null-additive subset of ${}^{\omega}2$ is constructed.

1. The basic definitions and the main theorem

1. *Definition:* (1) We define addition on ${}^{\omega}2$ as addition modulo 2 on each component, i.e., if $x, y, z \in {}^{\omega}2$ and $x + y = z$ then for every n we have $z(n) = x(n) + y(n) \pmod 2$. (2) For $A, B \subseteq {}^{\omega}2$ and $x \in {}^{\omega}2$ we set $x + A =^{\mathrm{df}} \{x + y\colon y \in A\}$, and we define $A + B$ similarly. (3) We denote the Lebesgue measure on ${}^{\omega}2$ by μ. We say that $X \subseteq {}^{\omega}2$ is **null-additive** if for every $A \subseteq {}^{\omega}2$ which is null, i.e. $\mu(A) = 0$, $X + A$ is null too. (4) We say that $X \subseteq {}^{\omega}2$ is **meager-additive** if for every $A \subseteq {}^{\omega}2$ which is meager also $X + A$ is meager.

2. THEOREM: *Every null-additive set is meager-additive.*

3. *Outline and discussion:* Theorem 2 answers a question of Pawlikowski. It will be proved in Section 2. In Section 3 we shall present direct characterizations

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of the null-additive sets, and in Section 4 we shall do the same for the meager-additive sets.

It is obvious that every countable set is both null-additive and meager-additive. Are there uncountable null-additive sets, and even null-additive sets of cardinality $2^{\aleph_0}$? It will be shown in Section 5 that if the continuum hypothesis holds, then there is such a set. Haim Judah has shown that there is a model of ZFC in which all the null-additive sets are countable, but there are in it uncountable meager-additive sets. This is the model obtained by adding to L more than $\aleph_1$ Cohen reals. In this model the Borel conjecture holds, and therefore every null-additive set is strongly meager and hence countable. On the other hand, in this model the uncountable set of all constructible reals is meager-additive.

2. The proof of Theorem 2

4. *Notation:* (1) we shall use variables as follow: i, j, k, l, m, n for natural numbers, f, g, h for functions from ω to ω, $\eta, \zeta, \nu, \sigma, \tau$ for finite sequences of 0's and 1's, x, y, z for members of ${}^{\omega}2$, A, B, X, Y for subsets of ${}^{\omega}2$, and S, T for trees. (2) ${}^{\omega>}2 = \bigcup_{n<\omega} {}^{n}2$. We shall denote subsets of ${}^{\omega>}2$ by U, V. For $\eta \in {}^{\omega>}2$, $U \subseteq {}^{\omega>}2$ and $x \in {}^{\omega}2$ we shall write $\eta + x$ for $\eta + x \restriction \text{length}(\eta)$, and $U + x$ for $\{\eta + x \colon \eta \in U\}$. (3) For $\eta, \nu \in {}^{\omega>}2$ we write $\eta \trianglelefteq \nu$ if ν is an extension of η. (4) A tree for us is a nonempty subset of ${}^{\omega>}2$ such that

(a) if $\eta \trianglelefteq \nu$ and $\nu \in T$ then also $\eta \in T$, and

(b) if $\eta \in T$ and $n > \text{length}(\eta)$ then there is a ν of length n such that $\eta \trianglelefteq \nu$ and $\nu \in T$.

(5) For a tree T, $\text{Lim}(T) = \{x \in {}^{\omega}2\colon \text{for every } n < \omega \text{ we have } x \restriction n \in T\}$. (6) A tree T is said to be **nowhere dense** if for every $\eta \in T$ there is a $\tau \in {}^{\omega>}2$ such that $\eta \trianglelefteq \tau$ and $\tau \notin T$. A set $B \subseteq {}^{\omega}2$ is nowhere dense if $B \subseteq \text{Lim}(T)$ for some nowhere dense tree T. (7) For every $x, y \in {}^{\omega}2$ we write $x \equiv y$ if $x(n) = y(n)$ for all but finitely many $n < \omega$. For $A \subseteq {}^{\omega}2$, $A^{\text{fin}} =^{\text{df}} \{y \in {}^{\omega}2\colon y \equiv x \text{ for some } x \in A\}$. (8) $U^{[\nu]} =^{\text{df}} \{\tau \in U\colon \tau \trianglelefteq \nu \text{ or } \nu \trianglelefteq \tau\}$ (read: U through ν). (9) $U^{\langle\nu\rangle} =^{\text{df}} \{\tau \in {}^{\omega>}2\colon \nu \frown \tau \in U\}$ (read: U above ν), and for $\eta \in {}^{\omega>}2$ we define $\eta^{\langle k\rangle} =^{\text{df}} \langle \eta(k+i)\colon i < \text{length}(\eta) - k\rangle$. (10) For $\nu, \eta \in {}^{\omega>}2 \cup {}^{\omega}2$ we write $\nu \sim_n \eta$ if $\text{length}(\nu) = \text{length}(\eta)$ and $\nu(i) = \eta(i)$ for every $n \le i < \text{length}(\nu)$. For $S \subseteq {}^{\omega>}2 \cup {}^{\omega}2$ we define $S^{\sim n} = \{\nu\colon \nu \sim_n \eta \text{ for some } \eta \in S\}$.

5. *Outline of the proof:* Let $X \subseteq {}^{\omega}2$ be null-additive. It clearly suffices to prove that for every $A \subseteq {}^{\omega}2$ which is nowhere dense, $X + A$ is meager. Given a nowhere dense tree S we shall give a condition which is, as we prove in Lemma 6, sufficient for a tree T to be such that $T + S$ is nowhere dense. Then we shall split X into a union $X = \bigcup_{i=1}^{\infty} X_i$ such that, for each i, $X_i \subseteq \mathrm{Lim}\,(T_i)$ where T_i is a tree which satisfies that condition. Thus for a nowhere dense S, each set $X_i + \mathrm{Lim}\,(S) \subseteq \mathrm{Lim}\,(T_i + S)$ is nowhere dense, hence $X + \mathrm{Lim}\,(S) \subseteq \bigcup_{i=1}^{\infty} \mathrm{Lim}\,(T_i + S)$ is meager.

6. LEMMA: *Let T be a tree such that*

(a) *T is nowhere dense.*

(b) *$f = f_T$ is the function from ω to ω given by $f(n) = \min\{m :$ for every $\eta \in {}^{n}2$ there is a $\tau \in {}^{m}2$ such that $\eta \trianglelefteq \tau$ and $\tau \notin T\}$. Thus for every sequence η of length n there is a witness of length $\leq f(n)$ exemplifying that T is nowhere dense. Obviously, for every $n < \omega$, $f(n) > n$, and if $n < m$, then $f(n) \leq f(m)$.*

Let g be a function from ω to ω. We can find $\overline{n} = \langle n_i \colon i < \omega\rangle$ and $\overline{n}' = \langle n'_i \colon i < \omega\rangle$, increasing sequences of natural numbers such that

(c) *$f^{g(i)}(n_i) \leq n'_i < n_{i+1}$ for every $i < \omega$, where f^m denotes the m-th iteration of f.*

Then for every tree S which satisfies

(d) *S is of width $(\overline{n}', g)$, i.e., for every $i < \omega$ we have $|{}^{n'_i}2 \cap S| \leq g(i)$,*

$T + S$ is nowhere dense.

Proof: Let $\eta \in {}^{n_i}2$. We shall show the existence of an $\eta' \in {}^{n'_i}2$ such that $\eta \trianglelefteq \eta'$ and $\eta' \notin T + S$.

By (c) there is a sequence $m_0, \ldots, m_{g(i)}$ such that $m_0 = n_i$, $f(m_k) \leq m_{k+1}$ for $0 \leq k \leq g(i)$ and $m_{g(i)} = n'_i$. Let $\langle \tau_k \colon k < k_i\rangle$ enumerate the set ${}^{n'_i}2 \cap S$. By (d), $k_i \leq g(i)$. We define $\eta_k \in {}^{m_k}2$ for $0 \leq k \leq k_i$ by recursion as follows. Start with $\eta_0 = \eta$. Given $\eta_k \in {}^{m_k}2$, for $k < k_i$, we shall define $\eta_{k+1} \in {}^{m_{k+1}}2$ so that for no extension $\eta' \in {}^{n'_i}2$ of η_{k+1} shall we have $\eta' + \tau_k \in T$. We have $\eta_k + \tau_k \restriction m_k \in {}^{m_k}2$ and, by the definition of f and by the choice of the m_k's, $\eta_k + \tau_k \restriction m_k$ has an extension $\nu \in {}^{m_{k+1}}2$ such that $\nu \notin T$. If we take $\eta_{k+1} = \nu + \tau_k \restriction m_{k+1}$ then $\eta_k + \tau_k \restriction m_k \trianglelefteq \nu$ implies $\eta_k \trianglelefteq \eta_{k+1}$, $\eta_{k+1} \in {}^{m_{k+1}}2$ and $\eta_{k+1} + \tau_k \restriction m_{k+1} = \nu \notin T$, and therefore for every $\eta' \in {}^{n'_i}2$ such that $\eta_k \trianglelefteq \eta'$ we have $\eta' + \tau_k \notin T$. Let $\eta' = \eta_{k_i}$, and assume that $\eta' \in T + S$. Then, for some $k < k_i \leq g(i)$, we have $\eta' + \tau_k \in T$, contradicting our choice of $\eta_{k+1} = \eta' \restriction m_{k+1}$. Thus $\eta' \notin T + S$.

7. LEMMA: *If S, T_i, $i \in \omega$ are trees and* $\mathrm{Lim}(S) \subseteq \bigcup_{i\in\omega} \mathrm{Lim}(T_i)$, *then for some* $\eta \in S$ *and* $j \in \omega$, $S^{[\eta]} \subseteq T_j$.

Proof: Suppose that this is not the case, i.e., for every $\eta \in S$ and $i < \omega$ there is a ζ such that $\zeta \in S^{[\eta]}$ and $\zeta \notin T_i$. Once there is such a ζ we can assume that $\eta \trianglelefteq \zeta$ and length$(\zeta) >$ length(η). We define now, by induction on i, η_i and k_i so that $k_i = \mathrm{length}(\eta_i)$, $k_0 = 0$, $\eta_0 = \langle\rangle$, $\eta_i \trianglelefteq \eta_{i+1}$, $k_i < k_{i+1}$, $\eta_{i+1} \in S$ and $\eta_{i+1} \notin T_i$. Let $y = \bigcup_{i\in\omega} \eta_i$. Since $\eta_i \in S$ for every $i \in \omega$, $y \in \mathrm{Lim}(S) \subseteq \bigcup_{i\omega} \mathrm{Lim}(T_i)$, hence for some $j \in \omega$, $y \in \mathrm{Lim}(T_j)$. However, $y \restriction k_{j+1} = \eta_{j+1} \notin T_j$, contradicting $y \in \mathrm{Lim}(T_j)$.

8. LEMMA: *Let S and T be trees such that* $\mathrm{Lim}(S) \subseteq (\mathrm{Lim}(T))^{\mathrm{fin}}$. *Then there are* $k < \omega$, $\eta, \nu \in {}^k2$, $\eta \in S$ *such that* $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$.

Proof: For $n < \omega$, $\sigma_1, \sigma_2 \in {}^n2$ and $\sigma_2 \in T$ we define

$$T_{\sigma_1,\sigma_2} =^{\mathrm{df}} \{\tau : \tau \trianglelefteq \sigma_1\} \cup \{\sigma_1 \frown \tau : \sigma_2 \frown \tau \in T\}.$$

(This is the tree $T^{[\sigma_2]}$ with "σ_2 replaced by σ_1".) Clearly

$$(\mathrm{Lim}(T))^{\mathrm{fin}} = \bigcup_{n<\omega,\, \sigma_1,\sigma_2 \in {}^n2,\, \sigma_2 \in T} \mathrm{Lim}(T_{\sigma_1,\sigma_2}). \tag{1}$$

Since there are only countably many T_{σ_1,σ_2}'s in (1), there are, by Lemma 7, a $\zeta \in S$ and $j < \omega$ such that $S^{[\zeta]} \subseteq T_{\sigma_1,\sigma_2}$. Clearly there is an η with $\zeta \trianglelefteq \eta$ and a ν with length$(\nu) =$ length(η) such that $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$. (If $\zeta \trianglelefteq \sigma_1$ then $\eta = \sigma_1$ and $\nu = \sigma_2$, else $\sigma_1 \trianglelefteq \zeta$ and then $\eta = \zeta$ and $\nu = \sigma_2 \frown \zeta \restriction [\mathrm{length}(\zeta), \mathrm{length}(\sigma_2))$.)

9. LEMMA: *Let X be a null-additive set. Let T be a tree such that* $\mu(\mathrm{Lim}(T)) > 0$. *There is a tree T^* such that* $\mu(\mathrm{Lim}(T^*)) > 0$, *moreover for every* $\eta \in T^*$ *also* $\mu(\mathrm{Lim}(T^{*[\eta]})) > 0$, *and* $(({}^\omega2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}}) + X) \cap \mathrm{Lim}(T^*) = \emptyset$, *and then*

$$X = \bigcup_{\eta \in T^*,\ \mathrm{length}(\zeta) = \mathrm{length}(\eta)} Y^X_{\eta,\zeta}$$

where $Y^X_{\eta,\zeta} = \{x \in X : \zeta \frown x^{\langle \mathrm{length}(\zeta)\rangle} + T^{*[\eta]} \subseteq T\}$.

Proof: Since $\mu(\mathrm{Lim}(T)) > 0$ then, as easily seen, $\mu((\mathrm{Lim}(T))^{\mathrm{fin}}) = 1$, hence $\mu({}^\omega2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}}) = 0$. Since X is null-additive also $\mu(X + ({}^\omega2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}}))$ $= 0$. Hence there is a tree T^* such that

$$\mu(\mathrm{Lim}(T^*)) > 0 \quad \text{and} \quad (X + ({}^\omega2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}})) \cap \mathrm{Lim}(T^*) = \emptyset.$$

Without loss of generality we can assume that T^* has been pruned so that, for $\eta \in T^*$, $\mu(\mathrm{Lim}\,(T^{*[\eta]})) > 0$.

Let $x \in X$; then

$$^{\omega}2 \setminus (x + (\mathrm{Lim}\,(T))^{\mathrm{fin}}) = x + (^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}) \subseteq X + (^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}).$$

Hence $(^{\omega}2 \setminus (x + (\mathrm{Lim}\,(T))^{\mathrm{fin}})) \cap \mathrm{Lim}\,(T^*) \subseteq (X + (^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}) \cap \mathrm{Lim}\,(T^*) = \emptyset$, i.e., $\mathrm{Lim}\,(T^*) \subseteq x + (\mathrm{Lim}\,(T))^{\mathrm{fin}}$, and therefore $\mathrm{Lim}\,(x + (T^*)) = x + \mathrm{Lim}\,(T^*) \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}}$. By Lemma 8 there are $\eta \in T^*$ and $\nu \in {}^{\mathrm{length}\,(\eta)}2$ such that $x^{\langle \mathrm{length}\,(\eta) \rangle} + T^{*\langle \eta \rangle} \subseteq T^{\langle \nu \rangle}$. Let $\zeta = \eta + \nu$; then $\zeta + \eta = \nu$ and therefore $\zeta \frown x^{\langle \mathrm{length}\,(\eta) \rangle} + T^{*[\eta]} \subseteq T^{[\nu]} \subseteq T$, hence $x \in Y^X_{\eta,\zeta}$.

10. Lemma: *Let X be null-additive, and let $\overline{n} = \langle n_i : i < \omega \rangle$, $\overline{n}' = \langle n'_i : i < \omega \rangle$ be such that, for every $i < \omega$, $n_i < n'_i$ and $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$; then we can represent X as $\bigcup_{m<\omega} X_m$ such that, for each m, for some real $a_m \in (0,1)$ and S_m of width $(\overline{n}', g_{a_m})$ we have $X_m \subseteq \mathrm{Lim}\,(S_m)$, where for every real $a \in (0,1)$, g_a is the function on ω given by $g_a(0) = 1$, $g_a(i) = \max(1, \mathrm{int}(\log_2(a)/\log_2(1-2^{-i})))$ for $i > 0$, and for a real d, $\mathrm{int}(d)$ is the integral part of d.*

Proof: Since $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$ we can fix for each $0 < i < \omega$ a sequence $\langle u_{i,\tau} : \tau \in {}^{n'_i}2 \rangle$ of pairwise disjoint subsets of the interval $[n'_i, n_{i+1})$ having i members each. Let $B \subseteq {}^{\omega}2$ be given by

$$B = \{y \in {}^{\omega}2 : (\forall j > 0)(\exists k \in u_{j, y \restriction n'_j})\, y(k) = 1\}.$$

B is clearly a closed subset of ${}^{\omega}2$, hence for $T = \{y \restriction n : y \in B \wedge n \in \omega\}$ $B = \mathrm{Lim}\,(T)$.

The properties of T in which we are interested are

(B0) $T \supseteq {}^{n_1}2$.

(B1) For each $\eta \in T \cap {}^{n'_i}2$ we have $\left|T^{[\eta]} \cap {}^{n_{i+1}}2\right| = 2^{(n_{i+1} - n'_i)}(1 - 2^{-i})$.

(B2) If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n'_i}2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu_l^+$ for $l < k$ and $\nu_0, \ldots, \nu_{k-1}$ is with no repetitions, then

$$\left|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}\right| \leq 2^{n_{i+1} - n'_i}\left(1 - 2^{-i}\right)^k.$$

(B3) For every $\eta \in {}^{n'_i}2$ we have: $\eta \restriction n_i \in T$ implies $\eta \in T$.

These properties can be established by an obvious counting argument.

By (B0), (B1) and (B3) we have

$$\begin{aligned}\mu(\mathrm{Lim}\,(T)) &= \mu\left(\bigcap_{i=1}^{\infty}\{x \in {}^{\omega}2\colon x \restriction n_i \in T\}\right)\\ &= \mu\left(\{x \in {}^{\omega}2\colon x \restriction n_1 \in T\}\right) \cdot \prod_{i=1}^{\infty} \frac{\mu(\{x \in {}^{\omega}2\colon x \restriction n_{i+1} \in T\})}{\mu(\{x \in {}^{\omega}2\colon x \restriction n_i \in T\})}\\ &= 1 \cdot \prod_{i=1}^{\infty} \frac{|T \cap {}^{n_{i+1}}2|/2^{n_{i+1}}}{|T \cap {}^{n_{i+1}}2|/2^{n_i}} = \prod_{i=1}^{\infty}(1-2^{-i}) > 0.\end{aligned}$$

For the T which we constructed and the given X, let T^* and $Y_{\eta,\zeta} = Y^X_{\eta,\zeta}$ be as in Lemma 9. For $\rho \in {}^{\mathrm{length}\,(\eta)}2$ let $Y_{\eta,\zeta,\rho} = \{y \in Y_{\eta,\zeta}\colon y \restriction \mathrm{length}\,(\eta) = \rho\}$. Clearly

$$X = \bigcup_{\eta \in T^*,\ \mathrm{length}\,(\eta)=\mathrm{length}\,(\zeta)=\mathrm{length}\,(\rho)} Y_{\eta,\zeta,\rho}. \tag{2}$$

Since there are only countably many $Y_{\eta,\zeta,\rho}$'s they can be taken to be the X_m's we are looking for, provided we show that every such $Y_{\eta,\zeta,\rho}$ is a subset of $\mathrm{Lim}\,(S)$ for some tree S of width $\langle \overline{n}', g_a\rangle$ for some real $0 < a < 1$. We shall see that this is indeed the case if we take $S = \{y \restriction m\colon y \in Y_{\eta,\zeta,\rho},\ m < \omega\}$ and $a = \mu(T^{*[\eta]})$; $a > 0$ by what we assumed about T^*. As, obviously, $Y_{\eta,\zeta,\rho} \subseteq \mathrm{Lim}\,(S)$, all we have to do is to show that S is of width $\langle \overline{n}', g_a\rangle$. We fix a $j \in \omega$.

We can choose a set $W \subseteq S \cap {}^{n_{j+1}}2$ such that the function mapping $\eta \in W$ to $\eta \restriction n'_j$ is one to one and onto $S \cap {}^{n'_j}2$.

We fix now η, ζ, ρ and denote $Y_{\eta,\zeta,\rho}$ by Y and the length of η, ζ, ρ by n. Let $z \in {}^{\omega}2$ be such that $z \restriction n = \zeta + \rho$ and $z(i) = 0$ for $i \geq n$. Then for every y such that $y \restriction n = \rho$ we have $y + z = \zeta \frown y^{\langle n\rangle}$. Therefore, by the definition of Y we have

$$\begin{aligned} Y &= \{y \in {}^{\omega}2\colon y \restriction n = \rho,\ (\zeta \frown y^{\langle n\rangle}) + T^{*[\eta]} \subseteq T\}\\ &= \{y \in {}^{\omega}2\colon y \restriction n = \rho,\ y + z + T^{*[\eta]} \subseteq T\};\end{aligned} \tag{3}$$

for every $y \in Y$ there is a unique $\tau \in W$ such that $\tau \restriction n'_j = y \restriction n'_j$ (τ may be $y \restriction n_{j+1}$). Clearly $|W| = |S \cap {}^{n'_j}2|$ and we denote $|W|$ by s, so it suffices to prove $s \leq g_a(j)$. If $n'_j \leq n$, then the only member of $S \cap {}^{n'_j}2$ is $\rho \restriction n'_j$ hence $s = 1$, so $s \leq g_a(j)$. We shall now deal with the case where $n'_j > n$. Let $\tau_0, \ldots, \tau_{s-1}$ be the members of W. For $m < s$, $\tau_m = y \restriction n_{j+1}$ for some $y \in Y$, hence, by (3),

$\tau_m + z + T^{*[\eta]} \subseteq T$ and therefore $(z + T^{*[\eta]}) \cap {}^{n_{j+1}}2 \subseteq \tau_m + T$. Since this holds for every $\tau \in W$ we have

$$z + T^{*[\eta]} \cap {}^{n_{j+1}}2 \subseteq \bigcap_{m<s} \tau_m + T. \tag{4}$$

Let us find out the size of $\bigcap_{m<s}(\tau_m + T)$. Let $\sigma \in {}^{n'_j}2$, and we shall ask how many members τ of $\bigcap_{m<s}(\tau_m + T)$ extend σ. Now $\tau \in \tau_m + T$ for each $m < s$ iff $\tau + \tau_m \in T$ for each $m < s$. If, for some $m < s$, $\sigma + \tau_m \restriction n'_j \notin T$, then also $\tau + \tau_m \notin T$, hence σ has no extension in $\bigcap_{m<s}(\tau_m + T)$. If, for every $m < s$, $\sigma + \tau_m \restriction n'_j \in T$, then by (B2) (where $\eta = \sigma$, $\nu_m = \tau_m \restriction n'_j$ and $\nu_m^+ = \tau_m$), since $\tau_m \restriction n'_j \neq \tau_l \restriction n'_j$ for $m \neq l$, the number of τ's such that $\sigma \trianglelefteq \tau \in {}^{n_{j+1}}2$ and $\tau + \tau_m \in T$ for every $m < s$ is $2^{n_{j+1} - n'_j}(1 - 2^{-j})^s$. Since there are $2^{n'_j}$ different σ's in ${}^{n'_j}2$ we have

$$\left| \bigcap_{m<s} (\tau_m + T) \right| \leq 2^{n_{j+1}} \cdot (1 - 2^{-j})^s. \tag{5}$$

On the other hand, since $\mu(T^{*[\eta]}) = a$, $T^{*[\eta]} \cap {}^{n_{j+1}}2$ has at least $a \cdot 2^{n_{j+1}}$ members, and so has $z + T^{*[\eta]} \cap {}^{n_{j+1}}2$. Comparing (4) with (5) we get $a \cdot 2^{n_{j+1}} \leq 2^{n_{j+1}}(1 - 2^{-j})^s$, i.e., $a \leq (1 - 2^{-j})^s$, $\log_2(a) \leq s \cdot \log_2(1 - 2^{-j})$, $s \leq \log_2(a)/\log_2(1 - 2^{-j})$.

11. *Proof of Theorem 2:* Let X be null-additive. As mentioned in subsection 5, it suffices to show that for every nowhere dense tree T, $X + \mathrm{Lim}\,(T)$ is meager. Let $f = f_T$ as in Lemma 6. Define by recursion $n_0 = 0$, $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1$ and $n_{i+1} = n'_i + i \cdot 2^{n'_i} + 1$. By Lemma 10, $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$, where for some $a_m \in (0,1)$ S_m is of width $\langle \overline{n}', g_{a_m} \rangle$, hence it suffices to show that if S is of width $\langle \overline{n}', g_a \rangle$ for some $a \in (0,1)$ then $\mathrm{Lim}\,(S) + \mathrm{Lim}\,(T) = \mathrm{Lim}\,(S + T)$ is meager. Let j be such that $\frac{1}{j+1} \leq a$ and let $\eta_1, \ldots, \eta_k$ be all the members of S of length n'_j. Then $S = \bigcup_{l=1}^k S^{[\eta_l]}$ and $\mathrm{Lim}\,(S) = \bigcup_{l=1}^k \mathrm{Lim}\,(S^{[\eta_l]})$. Therefore it suffices to prove that for $1 \leq l \leq k$, $\mathrm{Lim}\,(S_l) + \mathrm{Lim}\,(T)$ is meager and this follows once we show that $S_l + T$ is nowhere dense. To prove this we show that the requirements of Lemma 6 hold here for S_l, T. (a) and (b) hold by our choice of T and f. Let g be defined by $g(i) = 1$ for $i < j$ and $g(i) = g_a(i)$ for $i \geq j$. Now we shall see that (c) holds. For $i < j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $f(n) \geq n$ for every n, and for $i \geq j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f^{g_a(i)}(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $a \geq \frac{1}{j+1} \geq \frac{1}{i+1}$ and the map $a \mapsto g_a(i)$ is a

decreasing function of a. Thus for every $i < \omega$, $f^{g(i)}(n_i) \le f^{g_{1/(i+1)}(i)}(n_i) \le n'_i$. (d) of Lemma 6 holds since, for $i < j$, $|{}^{n'_i}2 \cap S_l| = 1 = g(i)$ and, for $i \ge j$, $|{}^{n'_i}2 \cap S_l| \le |{}^{n'_i}2 \cap S| \le g_a(i) = g(i)$.

3. Characterization of the null-additive sets

12. Definition: By a **corset** we mean a non-decreasing function f from ω to $\omega \setminus \{0\}$ which converges to infinity (i.e., for every $n < \omega$, $f(m) > n$ for all sufficiently large m). For a corset f, we say that a tree T is **of width** f if for every $n < \omega$, $|T \cap {}^n2| \le f(n)$; and we say that T is **almost of width** f if $|T \cap {}^n2| \le f(n)$ for all sufficiently large n.

13. THEOREM: *For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:*

a. *X is null-additive.*

b. *For every corset f there is a tree S of width f such that $X \subseteq (\mathrm{Lim}\,(S))^{\mathrm{fin}}$.*

c. *For every corset f there are trees S_m, $m < \omega$, which are almost of width f such that $X \subseteq \bigcup_{m<\omega} (\mathrm{Lim}\,(S_m))^{\mathrm{fin}}$.*

d. *For every corset f there are trees S_m, $m < \omega$, of width f such that $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$.*

Proof: (b)→(c) is obvious.

(c)→(d): Let S be a tree almost of width f. Then for some k we have $|T \cap {}^n2| \le f(n)$ for all $n \ge k$. By (1) of Lemma 8,

$$(\mathrm{Lim}\,(S))^{\mathrm{fin}} = \bigcup_{\sigma_1,\sigma_2 \in {}^k2,\, \sigma_2 \in S} \mathrm{Lim}\,(S_{\sigma_1,\sigma_2}).$$

Each S_{σ_1,σ_2} is of width f since, for $n \le k$, we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| = 1$ and for $n > k$ we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| \le |S \cap {}^n2| \le f(n)$. Therefore, if $X \subseteq \bigcup_{m<\omega} (\mathrm{Lim}\,(S_n))^{\mathrm{fin}}$ as in (c) then each S_m can be replaced by countably many S_{σ_1,σ_2}'s and (d) holds.

(d)→(b): Let f be a corset. We can easily define by recursion a sequence $0 = n_0 < n_1 < \cdots$ of natural numbers and a corset f^* such that for all $j < \omega$ and $m \ge n_{j+1}$ we have $(j+1) \cdot 2^{n_j} \cdot f^*(m) \le f(m)$.

For a given corset f, if X satisfies (d) let S^*_m, $m < \omega$, be as in (d) for the corset f^*. We construct now a set $S \subseteq {}^{\omega>}2$ by defining $S \cap {}^m2$ by recursion on m. $S \cap {}^02 = \{\langle\rangle\}$. For $n_i < m \le n_{i+1}$ let

$$S \cap {}^m2 = \{\eta \in {}^m2 \colon \eta \restriction n_i \in S \cap {}^{n_i}2 \text{ and } \eta \in S_j^{*\sim n_j} \text{ for some } j < i \vee j = 0\}.$$

S can be easily seen to be a tree, and clearly $(\mathrm{Lim}\,(S))^{\mathrm{fin}} \supseteq \bigcup_{n<\omega} \mathrm{Lim}\,(S^*_n) \supseteq X$. For $m \leq n_1$ easily $|S \cap {}^m 2| \leq f(m)$, for $n_i \leq m < n_{i+1}, i \geq 1$ we have $|S \cap {}^m 2| \leq \sum_{j\leq i} |S_j^{*\frown n_j} \cap {}^m 2| = \sum_{j \leq i} 2^{n_j} |S^*_j \cap {}^m 2| \leq (i+1) \cdot 2^{n_j} \cdot f^*(m) \leq f(m)$, thus S is of width f.

(d)→(a): Assume now that (d) holds for X, and let $A \subseteq {}^\omega 2$, $\mu(A) = 0$; we shall prove that $\mu(X + A) = 0$. First we shall mention two lemmas from measure theory the proof of which is left to the reader.

LEMMA A: *For every tree T with $\mu(\mathrm{Lim}\,(T)) = a > 0$ and $\epsilon > 0$ there is an $N \in \omega$ such that for every $n \geq N$ there is a $t \subseteq {}^n 2 \cap T$ such that $|t| \geq 2^n(a - \epsilon)$ and, for each $\eta \in t$, $\mu(\mathrm{Lim}\,(T^{[\eta]})) > 2^{-n}(1 - \epsilon)$.*

Using Lemma A one can prove

LEMMA B: *For every tree T with $\mu(\mathrm{Lim}\,(T)) > 0$, every $\epsilon > 0$ and every sequence $\langle \epsilon_i : 0 < i < \omega \rangle$ of positive reals there is a subtree T' of T and an increasing sequence $\langle n_i : i < \omega \rangle$ of natural numbers such that $n_0 = 0$, $\mu(\mathrm{Lim}\,(T')) > \mu(\mathrm{Lim}\,(T)) - \epsilon$ and*

(6) *for $i > 0$ and every $\eta \in {}^{n_i}2 \cap T'$, $\mu(\mathrm{Lim}\,(T'^{[\eta]})) > 2^{-n_i}(1 - \epsilon_i)$.*

By basic measure theory $\mu(A^{\mathrm{fin}}) = 0$, so there is a tree T such that $\mu(\mathrm{Lim}\,(T)) > 0$ and $\mathrm{Lim}\,(T) \cap A^{\mathrm{fin}} = \emptyset$ hence $(\mathrm{Lim}\,(T))^{\mathrm{fin}} \cap A = \emptyset$. Given $\epsilon < \mu(\mathrm{Lim}\,(T))$ and $\langle \epsilon_i : i < \omega \rangle$ as in Lemma B we obtain a subtree T' of T as in that lemma with $\mu(\mathrm{Lim}\,(T')) > 0$. The union of sufficiently many "finite translates" of T', i.e., trees T'_{σ_1, σ_2} as in (1) of Lemma 8, is a tree T'' satisfying (6) with $\mu(\mathrm{Lim}\,(T'')) \geq \frac{1}{2}$. $(\mathrm{Lim}\,(T''))^{\mathrm{fin}} = (\mathrm{Lim}\,(T'))^{\mathrm{fin}} \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}}$ and hence $\mathrm{Lim}\,(T'') \cap A \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}} \cap A = \emptyset$. We take now $\epsilon_i = 1/4(i+1)^3$ and take T to be T'' and we get $\mu(\mathrm{Lim}\,(T)) \geq \frac{1}{2}$ and

(7) for $i > 0$ and every $\eta \in {}^{n_i}2 \cap T$, $\mu(\mathrm{Lim}\,(T^{[\eta]})) > 2^{-n_i}\left(1 - \frac{1}{4(i+1)^3}\right)$.

Let f be the corset given by $f(n) = i + 1$ for $n_i \leq n < n_{i+1}$. By (d) there are trees S_m of width f such that $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$. To show that $\mu(X + A) = 0$ it clearly suffices to show that, for every tree S of width f, $\mu(\mathrm{Lim}\,(S) + A) = 0$.

We define

$$T^* = \{\eta \in {}^{\omega>}2 : \nu + \eta \in T \text{ for every } \nu \in S \text{ of the same length as } \eta\}.$$

We do not show that T^* is a tree but obviously, if $\zeta \trianglelefteq \eta \in T^*$, then $\zeta \in T^*$, thus $\mathrm{Lim}\,(T^*)$ is defined. If $\mu(\mathrm{Lim}\,(T^*)) > 0$ then, by a well-known property of the measure, $\mu(\mathrm{Lim}\,(T^*)^{\mathrm{fin}}) = 1$, hence in order to prove $\mu(\mathrm{Lim}\,(S) + A) = 0$ it suffices to prove $(\mathrm{Lim}\,(S) + A) \cap (\mathrm{Lim}\,(T^*))^{\mathrm{fin}} = \emptyset$. Assume $y \in (\mathrm{Lim}\,(S) + A) \cap (\mathrm{Lim}\,(T^*))^{\mathrm{fin}}$. Since $y \in (\mathrm{Lim}\,(T^*))^{\mathrm{fin}}$ there is a $y' \in {}^{\omega}2$ such that $y'(n) = y(n)$ for all sufficiently big n's and $y' \in \mathrm{Lim}\,(T^*)$. Since $y \in \mathrm{Lim}\,(S) + A$ there is an $x \in \mathrm{Lim}\,(S)$ such that $y + x \in A$, hence $y + x \notin (\mathrm{Lim}\,(T))^{\mathrm{fin}}$, hence $y' + x \notin \mathrm{Lim}\,(T)$. Therefore, for some n, $y' \restriction n + x \restriction n \notin T$, hence, by the definition of T^*, $y' \restriction n \notin T^*$ contradicting $y' \in \mathrm{Lim}\,(T^*)$.

We still have to prove that $\mu(\mathrm{Lim}\,(T^*)) > 0$. We shall prove, by induction on i, that

$$n_i \le n \le n_{i+1} \to |(T \setminus T^*) \cap {}^{n}2| \le 2^n \cdot \sum_{j<i} \frac{1}{4(j+1)^2}. \tag{8}$$

Once we establish (8) we notice that since

$$\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*) = \bigcup_{n<\omega} \mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 \colon x \restriction n \in T^*\},$$

and the set $\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 \colon x \restriction n \in T^*\}$ is increasing with n hence

$$\begin{aligned}
\mu(\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*)) &= \lim_{n\to\infty} \mu(\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 : x \restriction n \in T^*\}) \\
&\le \lim_{n\to\infty} 2^{-n} |(T \setminus T^*) \cap {}^{n}2| \\
&\le \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{4(j+1)^2} \\
&= \sum_{j=0}^{\infty} \frac{1}{4(j+1)^2} = \frac{\pi^2}{24} < \frac{1}{2}
\end{aligned}$$

and since $\mu(\mathrm{Lim}\,(T)) \ge \frac{1}{2}\ \mu(\mathrm{Lim}\,(T^*)) > 0$.

To prove (8), assume now $n_i \le n \le n_{i+1}$. By the definition of T^*

$$\begin{aligned}
(T \setminus T^*) \cap {}^{n}2 =& \{\eta \in T \cap {}^{n}2 \colon (\exists \rho \in S \cap {}^{n}2)(\rho + \eta \notin T)\} \\
=& \{\eta \in T \cap {}^{n}2 \colon (\exists \rho \in S \cap {}^{n}2)(\eta \restriction n_i + \rho \restriction n_i \notin T)\} \\
& \cup \bigcup_{\rho \in S \cap {}^{n}2} \{\eta \in T \cap {}^{n}2 \colon \eta \restriction n_i + \rho \restriction n_i \in T \wedge \eta + \rho \notin T\} \\
\subseteq& \{\eta \in {}^{n}2 \colon \eta \restriction n_i \in T \setminus T^*\} \\
& \cup \bigcup_{\rho \in S \cap {}^{n}2} \{\eta \in {}^{n}2 \colon \eta + \rho \in \{\sigma \in {}^{n}2 \colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}\}.
\end{aligned}$$

Therefore

$$|(T \setminus T^*) \cap {}^{n}2| \leq 2^{n-n_i} |(T \setminus T^*) \cap {}^{n_i}2| + |S \cap {}^{n}2| |\{\sigma \in {}^{n}2 \colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}|.$$

For $i > 0$ we have, by the induction hypothesis,

$$|T \setminus T^* \cap {}^{n_i}2| \leq 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2}.$$

For $i = 0$ we have $(T \setminus T^*) \cap {}^{n_i}2 = \emptyset$ since $n_0 = 0$ and $\emptyset \in T^*$.

$$|S \cap {}^{n}2| \leq f(n) = i \quad \text{and} \quad |\{\sigma \in {}^{n}2 \colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}| \leq \frac{2^n}{4(i+1)^3},$$

by (7). Thus

$$|(T \setminus T^*) \cap {}^{n}2| \leq 2^{n-n_i} \cdot 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2} + (i+1) \cdot \frac{2^n}{4(i+1)^3} \leq 2^n \sum_{j<i+1} \frac{1}{4(j+1)^2}$$

which is what we had to show.

(a)$\rightarrow$(c): Most of the proof follows that of Lemma 10. We need also the following Lemma 14, which will be proved later. Let f be a corset.

14. LEMMA: *There is an infinite sequence $0 = n_0 < n_1 < n_2 < \cdots$ and a tree T such that for every $i \in \omega$ we have $f(n_{i+1}) > (i+1) \cdot 2^{i+1} + 1$ and*

(B1) *For each $\eta \in T \cap {}^{n_i}2$ we have $|T^{[\eta]} \cap {}^{n_{i+1}}2| = 2^{(n_{i+1}-n_i)} \cdot (1 - 2^{-(i+1)})$.*

(B2) *If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n_i}2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu_l^+$ for $l < k$, then*

$$\left|\{\eta^+ \colon \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}\right| \leq 2^{n_{i+1}-n_i} \left(1 - 2^{-(i+1)}\right)^{k-1}$$

Let $\langle n_i \colon i \in \omega \rangle$ and T be as in Lemma 14. As in the proof of Lemma 10 we get $\mu(\mathrm{Lim}\,(T)) > 0$. Let T^* and $Y_{\eta,\zeta} = Y^X_{\eta,\zeta}$ be as in Lemma 9 and let $Y_{\eta,\zeta,\rho}$, S and z be as in the proof of Lemma 10. All we have to do is to show that S is almost of width f. Let us fix η, ζ and ρ. We shall now see that

(9) If $\eta' \in T^{*[\eta]} \cap {}^{n_i}2$ then
$$|\{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\}| / 2^{(n_{i+1}-n_i)} \leq (1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2| - 1}.$$

Let $\eta^+ \in T^{*[\eta]} \cap {}^{n_{i+1}}2$; then, by the definition of S (see (3)), if $\rho^+ \in S \cap {}^{n_{i+1}}2$ then $\rho^+ + \eta^+ + z \in T$. Thus

$$\{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\} \subseteq \{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)\rho^+ + \eta^+ + z \in T\}.$$

Let us take in (B2) $\eta = \eta'$, $k = |S \cap {}^{n_i}2|$, $\{\tau_l \colon l < k\} = S \cap {}^{n_i}2$, $\{\tau_l^+ \colon l < k\} \subseteq S \cap {}^{n_{i+1}}2$, and, for $l < k$, $\tau_l^+ \restriction n_i = \tau_l$, $\nu_l = \tau_l + z$, $\nu_l^+ = \tau_l^+ + z$; hence $\nu_l = \nu_l^+ \restriction n_i$ for $l < k$. Since for $l < k$, $\nu_l^+ + z = \tau_l^+ \in S \cap {}^{n_{i+1}}2$ we have

$$\begin{aligned}&\{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\}\\ &\subseteq \{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\nu_l^+ + \eta^+ \in T)\},\end{aligned}$$

therefore by (B2)

$$\begin{aligned}&|\{\eta^+ \colon \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\}|\\ &\le 2^{n_{i+1}-n_i}(1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2|-1},\end{aligned}$$

which establishes (9).

(9) tells us how T^* grows from the level n_i to the level n_{i+1} and therefore $|T^* \cap {}^{n_i}2| \cdot 2^{-n_i} \le \prod_{j<i}(1 - 2^{-(j+1)})^{|S \cap {}^{n_j}2|-1}$. Let $c_0 = \mu(\operatorname{Lim} T^*)$. We know that $c_0 > 0$ and we can assume $c_0 < 1$. Then

$$-\infty < \log c_0 \le \log(|T^* \cap {}^{n_i}2| \cdot 2^{-n_i}) \le \sum_{j<i}\big(\log(1 - 2^{-(j+1)}) \cdot (|S \cap {}^{n_j}2| - 1)\big).$$

Since $\log(1-x) \le -\frac{1}{2}x$ we get

$$\sum_{j<i} 2^{-(j+2)} \cdot (|S \cap {}^{n_{i+1}}2| - 1) \le \log \frac{1}{c_0}.$$

We shall denote $4 \log \frac{1}{c_0}$ by c, so $\sum_{j<i} 2^{-j} \cdot (|S \cap {}^{n_j}2| - 1) \le c$, and for every j, $2^{-j}(|S \cap {}^{n_j}2| - 1) \le c$, hence $|S \cap {}^{n_j}2| \le c \cdot 2^j + 1$. For $j > c$ we have, by our choice of the n_i's, $f(n_j) > j \cdot 2^j + 1 > c \cdot 2^j + 1 \ge |S \cap {}^{n_j}2|$, hence S is almost of width f.

Lemma 14 follows immediately from the following Lemma.

15. Lemma: *For every $n \in \omega$ and $0 < p < 1$ there is an $N > n$ such that, for every $n' \ge N$ and $t \subseteq {}^n2$, there is a $t' \subseteq {}^{n'}2$ which satisfies the following (i)–(iii).*

(i) *For each $\zeta \in t'$, $\zeta \restriction n \in t$.*

(ii) *For each* $\eta \in t$, $|t'^{[\eta]}| \geq 2^{n'-n} \cdot p$.

(iii) *If* $0 < k \leq 2^n$, $\eta, \nu_0, \dots, \nu_{k-1} \in {}^n2$, $\nu_0^+, \dots, \nu_{k-1}^+ \in {}^{n'}2$, $\nu_j^+ \neq \nu_l^+$ *for* $j < l < k$, $\eta + \nu_l \in t$, $\nu_l = \nu_l^+ \restriction n$ *for* $l < k$, *then*

$$|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2, (\forall l < k)\, \eta^+ + \nu_l^+ \in t'\}| \leq 2^{n'-n} p^{k-1}.$$

Proof: We shall prove the lemma by the probabilistic method. Let $n' > n$ and let $A = \{\eta^+ \in {}^{n'}2 : \eta^+ \restriction n \in t\}$. We construct a subset A^* of A as follows. We take a coin which yields heads with probability p. For each $\eta^+ \in A$ we toss this coin and we put η^+ in A^* iff the coin shows heads. We shall see that if we take $t' = A^*$ then, for sufficiently large n', the probability that (ii) holds has a positive lower bound which does not depend on n' while the probability that (iii) holds is arbitrarily close to 1. Hence there is an N and a t' as claimed by the lemma. We prove first two lemmas.

LEMMA 16: *For* $k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+$ *as in Lemma 15 there are reals* $c_1, c_2 > 0$ *which depend only on* p, n *and* k *such that*

$$\Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2, \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A*\}| \geq p^{k-1} 2^{n'-n}\Big) < c_1 e^{-c_2 \cdot 2^{n'}}.$$

Proof: We denote $2^{n'-n}$ with m. We set $({}^{n'}2)^{[\eta]} = \{\eta_j^+ : j < m\}$. Let G be the graph on m given by

$$iGj \text{ iff } \{\eta_i^+ + \nu_l^+ : l < k\} \cap \{\eta_j^+ + \nu_l^+ : l < k\} \neq \emptyset.$$

Obviously, each $i < m$ has at most k^2 neighbors in G hence, by a well known theorem, m can be decomposed into $k^2 + 1$ pairwise disjoint sets $B_0, \dots, B_{k^2}$ such that, for every $i \leq k^2$, if $j, l \in B_i$ and $j \neq l$ then jGl does not hold. Let $d < \frac{1}{2}\min\{p^{l-1} - p^l : l \leq 2^n\} = \frac{1}{2} p^{2^n - 1}(1-p) > 0$.

$$\tag{10} \begin{aligned} &\Pr\Big(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \geq m \cdot p^{k-1}\Big) \\ &\leq \Pr\Big(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)\Big) \quad \text{since } p^k + d < p^{k-1}. \end{aligned}$$

Assume that

$$\tag{11} \begin{aligned} &\text{for every } i \leq k^2 \text{ such that } |B_i| \geq \frac{dm}{2k^2+2} \\ &\text{we have } |\{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \leq |B_i|\big(p^k + \frac{d}{2}\big); \end{aligned}$$

then

$$\{j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}$$
$$\subseteq \bigcup_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \quad \bigcup_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} B_i$$

hence

$$\begin{aligned}
&|j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \\
&\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} |j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| + \sum_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} |B_i| \\
&\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} |B_i|(p^k + \frac{d}{2}) + \sum_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} |B_i|, \qquad \text{by (11)} \\
&\le m(p^k + \frac{d}{2}) + (k^2+1)\frac{dm}{2k^2+2} = m(p^k + d).
\end{aligned}$$

Therefore the event $|\{j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)$ is incompatible with (11), so we continue the inequality (10) by

$$\begin{aligned}
&\le \Pr\Big(\bigvee_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} (|\{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2}))\Big) \\
&\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \Pr\Big(|\{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big).
\end{aligned} \tag{12}$$

For a fixed $j < m$ the events $\eta_j^+ + \nu_l^+ \in A^*$ for different l's are independent, hence $\Pr\left(\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\right) = p^k$. For a fixed i the events $\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*$ for different j's in B_i are independent since, by the definition of the B_i's, if $j_1, j_2 \in B_i$ and $j_1 \neq j_2$ then $\eta_{j_1}^+ + \nu_{l_1}^+ \neq \eta_{j_2}^+ + \nu_{l_2}^+$. We have here $|B_i|$ independent events, each with probability p^k. By a formula of probability theory (see, e.g., the formula $\Pr[X > a] < e^{-2a^2/n}$ in Spencer [2], p. 29)

$$\Pr\Big(\{j \in B_i\colon \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|p^k + \epsilon\Big) < e^{-\frac{2\epsilon^2}{|B_i|}}$$

and, taking $\epsilon = \frac{1}{2}|B_i|d$, we get

$$\Pr\Big(\{j \in B_i\colon \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big) < e^{-\frac{d^2|B_i|}{2}}.$$

Continuing (12) we get

$$\leq \sum_{i\leq k^2,\, |B_i|\geq \frac{dm}{2k^2+2}} e^{-\frac{d^2|B_i|}{2}} \leq \sum_{i\leq k^2,\, |B_i|\geq \frac{dm}{2k^2+2}} e^{-\frac{d^2}{2}\frac{dm}{2k^2+2}} \leq (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}}.$$

Combining this with the inequalities (10) and (12) we get

$$\begin{aligned}&\Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A^*\}| \geq p^{k-1}2^{n'-n}\Big)\\ &< (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}} = (k^2+1)e^{-\frac{d^3 2^{-n}2^{n'}}{4k^2+4}}.\end{aligned}$$

Since $d = \frac{1}{2}p^{2^n-1}(1-p)$ this proves Lemma 16.

17. LEMMA: *There are c_3, c_4 which depend only on p and n such that*

$$\begin{aligned}&\Pr\Big(\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ (13)\qquad &\qquad |\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1}\Big)\\ &\leq c_3(2^{n'})^{2^n}e^{-c_4 2^{n'}}\end{aligned}$$

where $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ are as in (iii) *of Lemma 15.*

Proof: By our requirements on $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ there are at most 2^n possible k's and η's and $(2^{n'})^{2^n}$ sequences $\langle \nu_0^+, \ldots, \nu_{k-1}^+\rangle$, while $\nu_0, \ldots, \nu_{k-1}$ are determined by $\nu_0^+, \ldots, \nu_{k-1}^+$ and n. Therefore we get, by Lemma 16,

$$\begin{aligned}&\Pr\Big(\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ &\qquad \big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1}\big)\Big)\\ &\leq \sum_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ &\qquad \Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1}\Big)\\ &\leq 2^n \cdot 2^n \cdot (2^{n'})^{(2^n)} \cdot c_1 e^{-c_2 2^{n'}}.\end{aligned}$$

Proof of Lemma 15 (continued): For each $\eta^+ \in {}^{n'}2$ such that $\eta \trianglelefteq \eta^+$, $\eta^+ \in A^*$ if the coin shows heads and different tosses are independent, $|A^{*[\eta]}|$ is a binomial random variable with expectation $2^{n'-n}p$. By the central limit theorem of probability theory (see, e.g., Feller [1, Ch. 7]) the limit, as $n' \to \infty$, of $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right)$ is $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx = \frac{1}{2}$, hence there is an N such that, for every $n' \geq N$, $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right) \geq \frac{1}{3}$. For different $\eta \in t$ the random variables $|A^{*[\eta]}|$ are idependent, hence

$$\Pr\Big(\bigwedge_{\eta\in t}(|A^{*[\eta]}| \geq 2^{n'-n}p)\Big) \geq \frac{1}{3^{|t|}} \geq \frac{1}{3^{2^n}}. \tag{14}$$

The right-hand side of (13) clearly vanishes as $n \to \infty$. Let us take N to be such that, for $n' \geq N$, the right-hand side of (13) is $< 3^{-2^n}$. Therefore we have, by (13) and (14),

$$\begin{aligned}\Pr\Big(&\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+} \\ &(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\,\eta^+ + \nu_l^+ \in A^*\}| < 2^{n'-n}p^{k-1}) \\ &\wedge \bigwedge_{\eta\in t}(|A^{*[\eta]}| \geq 2^{n'-n}p)\Big) > 0.\end{aligned} \tag{15}$$

By (15) there is a t' as required by the lemma.

4. Characterization of the meager-additive sets

18. THEOREM: *For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:*

(a) *X is meager additive.*

(b) *For every sequence $n_0 < n_1 < n_2 < \cdots$ of natural numbers there is a sequence $i_0 < i_1 < \cdots$ of natural numbers and a $y \in {}^\omega 2$ such that, for every $x \in X$ and for every sufficiently big $k < \omega$, there is an $l \in [i_k, i_{k+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$.*

Proof: Throughout this proof, if $x \in {}^\omega 2 \cup {}^{\omega>}2$, $k, l \in \omega$ and $k < l$, then $x \restriction [k, l)$ will denote the sequence $\xi \in {}^{l-k}2$ such that $\xi(i) = x(k+i)$ for all $i < l - k$.

(b)$\to$(a): In order to prove (a) it clearly suffices to show that $X + \mathrm{Lim}(T)$ is meager for every nowhere dense tree T.

For a nowhere dense tree T, let $\langle n_i : i < \omega\rangle$ be an ascending sequence of natural numbers such that $n_0 = 0$ and, for every $i \in \omega$, there is a sequence $\nu_i \in {}^{n_{i+1}-n_i}2$ such that, for every $\tau \in {}^{n_i}2$, $\tau \frown \nu_i \notin T$. Let $\langle i_j : j < \omega\rangle$ and y be as in (b); then, by (b), $X = \bigcup_{k\in\omega} X_k$ where

$$X_k = \{x \in X : (\forall m \geq k)(\exists l \in [i_m, i_{m+1}))\, x \restriction [n_l, n_{l+1}) = y \restriction (n_l, n_{l+1})\}.$$

It clearly suffices to prove that $X_k + \mathrm{Lim}\,(T)$ is nowhere dense.

Let $\tau \in {}^{n_{i_m}}2$ for some $m \geq k$; we shall show that τ has an extension which is not in $X_k + \mathrm{Lim}\,(T)$. Let $\nu = \nu_{i_m} \frown \nu_{i_m+1} \frown \cdots \frown \nu_{i_{m+1}-1}$ and let $\rho = y \restriction [n_{i_m}, n_{i_{m+1}}) + \nu$. We show that no extension z of $\tau \frown \rho$ is in $X_k + \mathrm{Lim}\,(T)$. Suppose $\tau \frown \rho \trianglelefteq z \in X_k + \mathrm{Lim}\,(T)$; then $z = x + w$, $x \in X_k$, $w \in \mathrm{Lim}\,(T)$. Therefore $\tau = \tau_1 + \tau_2$ and $\rho = \rho_1 + \rho_2$ such that $\tau_1 \frown \rho_1 \trianglelefteq x$ and $\tau_2 \frown \rho_2 \trianglelefteq w$, hence $\tau_2 \frown \rho_2 \in T$. Let $\xi \in {}^{n_{i_m}}2$ be such that $\xi(j) = 0$ for every $j < n_{i_m}$, and let $\rho' = \xi \frown \rho$, $\rho_1' = \xi \frown \rho_1$, $\rho_2' = \xi \frown \rho_2$. Clearly $\rho' = \rho_1' + \rho_2'$. Since $x \in X_k$ there is, by (b), an $l \in [i_m, i_{m+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$. Since $\tau_1 \frown \rho_1 \trianglelefteq x$ we have $\rho_1' \restriction [n_{i_m}, n_{i_m+1}) = x \restriction [n_{i_m}, n_{i_m+1})$ and hence $\rho_1 \restriction [n_l, n_{l+1}) = x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$. Therefore, by the definition of ρ and ν,

$$\begin{aligned} y \restriction [n_l, n_{l+1}) + \rho_2' \restriction [n_l, n_{l+1}) &= \rho_1' \restriction [n_l, n_{l+1}) + \rho_2' \restriction [n_l, n_{l+1}) = \rho' \restriction [n_l, n_{l+1}) \\ &= y \restriction [n_l, n_{l+1}) + \nu_l, \end{aligned}$$

hence $\rho_2' \restriction [n_l, n_{l+1}) = \nu_l$. By the definition of ν_l, $\tau_2 \frown \rho_2 \notin T$, contradicting $\tau_2 \frown \rho_2 \in T$.

(a)→(b): Let X be meager-additive. Let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers. Let $B = \{x \in {}^\omega 2 : \forall j (\exists k \in [n_j, n_{j+1}))\, x(k) \neq 0\}$ and $T = \{x \restriction n : x \in B, n \in \omega\}$. Clearly $B = \mathrm{Lim}\,(T)$ is nowhere dense, so $X + \mathrm{Lim}\,(T)$ is meager, hence there are nowhere dense trees S_n, $n \in \omega$ such that, for every n, $S_n \subseteq S_{n+1}$ and $X + \mathrm{Lim}\,(T) \subseteq \bigcup_{n \in \omega} S_n$. We define now $\langle i_l : l < \omega \rangle$, an ascending sequence of natural numbers, and $\langle \nu_l : l < \omega \rangle$, a sequence in ${}^{\omega >}2$, by recursion as follows. Let $i_0 = 0$. Given i_l let ν_l and i_{l+1} be such that $\nu_l \in {}^{n_{i_{l+1}} - n_{i_l}}2$ and, for every $\rho \in {}^{n_{i_l}}2$, $\rho \frown \nu_l \notin S_l$; there are such ν_l and i_{l+1} since S_l is nowhere dense. Let $y \in {}^\omega 2$ be given by $y \restriction [n_{i_l}, n_{i_{l+1}}) = \nu_l$ for every $l < \omega$. We shall now prove that $\langle i_l : l < \omega \rangle$ and y are as required by (b).

Let $x \in X$, so $\mathrm{Lim}\,(x + T) = x + \mathrm{Lim}\,(T) \subseteq X + \mathrm{Lim}\,(T) \subseteq \bigcup_{n \in \omega} S_n$. Therefore, by Lemma 7 (where we take $x + T$ for S) there is an $\eta \in T$ and $n \in \omega$ such that $x + T^{[\eta]} \subseteq S_n$. Let k be such that $k \geq n$ and $i_k \geq \mathrm{length}\,(\eta)$. By $x + T^{[\eta]} \subseteq S_n$ we have $x \restriction n_{i_{k+1}} + (T^{[\eta]} \cap {}^{n_{i_{k+1}}}2) \subseteq S_n \subseteq S_k$. Thus for every $\rho \in T^{[\eta]} \cap {}^{n_{i_{k+1}}}2$, $x \restriction n_{i_{k+1}} + \rho \in S_k$, hence, by the definition of ν_k and y,

$$x \restriction [n_{i_k}, n_{i_{k+1}}) + \rho \restriction [n_{i_k}, n_{i_{k+1}}) \neq \nu_k = y \restriction [n_{i_k}, n_{i_{k+1}})$$

and therefore $x \restriction [n_{i_k}, n_{i_{k+1}}) - y \restriction [n_{i_k}, n_{i_{k+1}}) \neq \rho \restriction [n_{i_k}, n_{i_{k+1}})$, i.e.,

$$x \restriction [n_{i_k}, n_{i_{k+1}}) - y \restriction [n_{i_k}, n_{i_{k+1}}) \notin \{\rho \restriction [n_{i_k}, n_{i_{k+1}}) \colon \rho \in T^{[\eta]}\}.$$

Since $i_k > \mathrm{length}(\eta)$ this can happen, by the definition of T, only if for some $i_k \leq j < i_{k+1}$, $x \restriction [n_j, n_{j+1}) - y \restriction [n_j, n_{j+1})$ is identically zero, and this is what we had to prove.

5. An uncountable null-additive set

19. THEOREM: *If the continuum hypothesis holds, then there is an uncountable null-additive set.*

Proof: Let, by CH, $\langle f_\alpha \colon \alpha < \omega_1 \rangle$ be a sequence containing all corsets and let $\langle T_\alpha \colon \alpha < \omega_1 \rangle$ be a sequence containing all perfect trees. Let E be the set of all limit ordinals $\delta < \omega_1$ such that, for every $\alpha, \beta < \delta$ and $n < \omega$, there is a $\gamma < \delta$ such that

$$T_\gamma \subseteq T_\alpha, \quad T_\gamma \cap {}^n2 = T_\alpha \cap {}^n2$$

$$\text{and, for all } m, \quad |T_\gamma \cap {}^m2| \leq \max(|T_\alpha \cap {}^m2|, f_\beta(m)).$$

Clearly E is closed. For every $\alpha, \beta < \omega_1$ there is a perfect tree T such that $T \subseteq T_\alpha$, $T \cap {}^n2 = T_\alpha \cap {}^n2$ and, for all $m < \omega$, we have $|T \cap {}^m2| \leq \max(|T_\alpha \cap {}^m2|, f_\beta(m))$. This tree T is T_γ for some $\gamma < \omega_1$. By a simple closure argument this implies that E is unbounded.

We need now the following lemma which will be proved later.

20. LEMMA: *There is an increasing and continuous sequence $\langle \delta_\zeta \colon \zeta < \omega_1 \rangle$ of ordinals in E such that for every $\zeta < \omega_1$, $k < \omega$ and $\alpha < \delta_\zeta$ there is an ordinal γ which is good for (ζ, α, k), where by γ **is good for** (ζ, α, k) we mean that*

(16)
(i) $\gamma < \delta_{\zeta+1}$,
(ii) $T_\gamma \subseteq T_\alpha$, $T_\gamma \cap {}^k2 = T_\alpha \cap {}^k2$,
(iii) *for all $\xi \leq \zeta$ such that $\delta_\xi > \alpha$ and for every $\epsilon < \delta_\zeta$, there is a $\beta < \delta_\xi$ such that $T_\gamma \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ.*

For $\xi < \omega_1$ let γ_ξ be the γ which is good for $(\xi, 0, 0)$. We choose $\eta_\xi \in \mathrm{Lim}(T_{\gamma\xi}) \setminus \{\eta_\beta \colon \beta < \xi\}$, and let $X = \{\eta_\xi \colon \xi < \omega_1\}$. X is clearly uncountable. We shall prove that X is null-additive by proving that X satisfies condition (c) of Theorem 13. For a given corset f, $f = f_\epsilon$ for some $\epsilon < \omega_1$. Let $\xi < \omega_1$ be such that $\delta_\xi > \epsilon$. Let $Z = \{\beta < \delta_{\xi+1}: T_\beta \text{ is almost of width } f_\epsilon\}$. We shall see that $X \subseteq \{\eta_\zeta: \zeta \leq \xi\} \cup \bigcup_{\beta \in Z} \operatorname{Lim}(T_\beta)$. Since Z and ξ are countable, condition (c) of Theorem 13 holds.

Let $\zeta > \xi$; it suffices to prove that $\eta_\zeta \in \operatorname{Lim}(T_\beta)$ for some $\beta \in Z$. $\epsilon < \delta_\xi$ and, since γ_ζ is good for $\alpha = k = 0$, there is a $\beta < \delta_\xi$ such that $T_{\gamma_\zeta} \subseteq T_\beta$, and T_β is of width f_ϵ. Thus $\beta \in Z$ and $\eta_\zeta \in \operatorname{Lim}(T_{\gamma\zeta}) \subseteq \operatorname{Lim}(T_\beta)$.

Proof of Lemma 20: We define $\langle \delta_\zeta: \zeta < \omega_1 \rangle$ as follows. δ_0 is the least member of E. For a limit ordinal ζ, we set $\delta_\zeta = \bigcup_{\xi<\zeta} \delta_\xi$. Since $\delta_\xi \in E$ for $\xi < \zeta$, also $\delta_\zeta \in E$. We shall now define $\delta_{\zeta+1}$. We shall assume, as an induction hypothesis, that for each $\xi < \zeta$ the lemma holds. For each $\alpha < \delta_\zeta$ and $k < \omega$ we shall find a $\gamma(\alpha, k)$ which is good for (ζ, α, k) and we shall choose $\delta_{\zeta+1}$ to be the least member of E greater than all these $\gamma(\alpha, k)$'s.

First we shall show that what the lemma claims holds for the case where ζ is a successor or 0. Whenever we shall write $\zeta - 1$ we shall assume that ζ is a successor. Let $\alpha < \delta_\zeta$ and $k < \omega$ be given, and let $\{\epsilon_n: n < \omega\} = \{\epsilon: \epsilon < \delta_\zeta\}$. We define sequences $\langle \alpha_n: n < \omega \rangle$ and $\langle k_n: n < \omega \rangle$ so that

(a) $k_0 = k$. If $\zeta = 0$ or $\alpha < \delta_{\zeta-1}$ then $\alpha_0 = \alpha$. If $\alpha \geq \delta_{\zeta-1}$ then α_0 is an ordinal which is good for $(\zeta - 1, \alpha, k)$. In any case $\alpha_0 < \delta_\zeta$, $T_{\alpha_0} \subseteq T_\alpha$ and $T_{\alpha_0} \cap {}^k2 = T_\alpha \cap {}^k2$.

(b) $\alpha_{n+1} < \delta_\zeta$.

(c) $T_{\alpha_{n+1}} \subseteq T_{\alpha_n}$.

(d) $T_{\alpha_{n+1}} \cap {}^{k_n}2 = T_{\alpha_n} \cap {}^{k_n}2$.

(e) $T_{\alpha_{n+1}}$ is almost of width f_{ϵ_n}.

(f) $k_{n+1} > k_n$ and every $\eta \in T_{\alpha_{n+1}} \cap {}^{k_n}2$ has at least two extensions in $T_{\alpha_{n+1}} \cap {}^{k_{n+1}}2$.

There are indeed such sequences $\langle \alpha_n: n < \omega \rangle$ and $\langle k_n: n < \omega \rangle$. (a) determines k_0 and α_0; if $\alpha < \delta_{\zeta-1}$, then there is an α_0 as in (a) by the induction hypothesis. δ_ζ is in E and let us take $\alpha_n, \epsilon_n, k_n, \alpha_{n+1}$ for α, β, n, γ in the definition of E, then $\delta_\zeta \in E$ says that there is an α_{n+1} which satisfies (b)–(e). Since $T_{\alpha_{n+1}}$ is perfect there is a k_{n+1} as in (f).

Let $T = \bigcap_{n\in\omega} T_{\alpha_n}$. By (c), (d), (f) T is a perfect tree, hence it is T_γ for some $\gamma < \omega_1$. Since T, and therefore also γ, depend on α and k, we denote γ by $\gamma(\alpha, k)$. As is easily seen $T_{\gamma(\alpha,k)} \subseteq T_\alpha$, $T_{\gamma(\alpha,k)} \cap {}^k2 = T_\alpha \cap {}^k2$, and, for every $\epsilon < \delta_\zeta$, $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{l+1}} \subseteq T_\alpha$, where l is such that $\epsilon = \epsilon_l$. This means that (iii) of (16) holds for $\xi = \zeta$. We shall have to show that (iii) holds for $\xi < \zeta$ and to deal with the case where ζ is a limit ordinal.

If ζ is a limit ordinal let $\langle \zeta_n : n < \omega \rangle$ be an increasing sequence such that $\delta_{\zeta_0} > \alpha$ and $\bigcup_{n<\omega} \zeta_n = \zeta$. We construct the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ as in the case where ζ is a successor, except that (a), (b), (e) are replaced by

(a′) $k_0 = k$, $\alpha_0 = \alpha$.

(b′) $\alpha_n < \delta_{\zeta_n}$.

(e′) α_{n+1} is good for (ζ_n, α, k).

By the induction hypothesis that the lemma holds for the ζ_n's there are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. Let $T = \bigcap_{n<\omega} T_{\alpha_n}$. As above, T is a perfect tree and $T = T_{\gamma(\alpha,k)}$, $T_{\gamma(\alpha,k)} \subseteq T_\alpha$ and $T_{\gamma(\alpha,k)} \cap {}^k 2 = T_\alpha \cap {}^k 2$.

We shall now see that for both cases of ζ with which we are dealing, (iii) holds for $\xi < \zeta$. If ζ is a successor then $\xi \leq \zeta - 1$ and, since α_0 is, by (a), good for $(\zeta - 1, \alpha, k)$, there is a $\beta < \delta_\zeta$ such that $T_{\alpha_0} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Note that if $\alpha < \delta_{\zeta-1}$ then, by the induction hypothesis, we have a $\gamma < \delta_\zeta$ which is good for (ζ, α, k), and if $\zeta = 0$ then (iii) holds vacuously, hence we may assume that $\zeta > 0$ and $\alpha \in [\delta_{\zeta-1}, \delta_\zeta)$. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_0}$, β is as required by (iii). If ζ is a limit ordinal, then $\xi \leq \zeta_n$ for some $n < \omega$. Since α_{n+1} is good for ζ_n, then there is a $\beta < \delta_\xi$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$, β is as required by (iii).

The only case left is that where ζ is a limit ordinal and $\xi = \zeta$ in (iii). Since $\alpha, \epsilon < \zeta$ also $\alpha, \epsilon < \zeta_n$ for some $n < \omega$. α_{n+1} is good for ζ_n, hence there is a $\beta < \delta_{\zeta_n}$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$ and $\zeta_n < \zeta$, β is as required by (iii).

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